Frequency Domain Gravitational Waveform Modelling for Eccentric Black Hole Binaries

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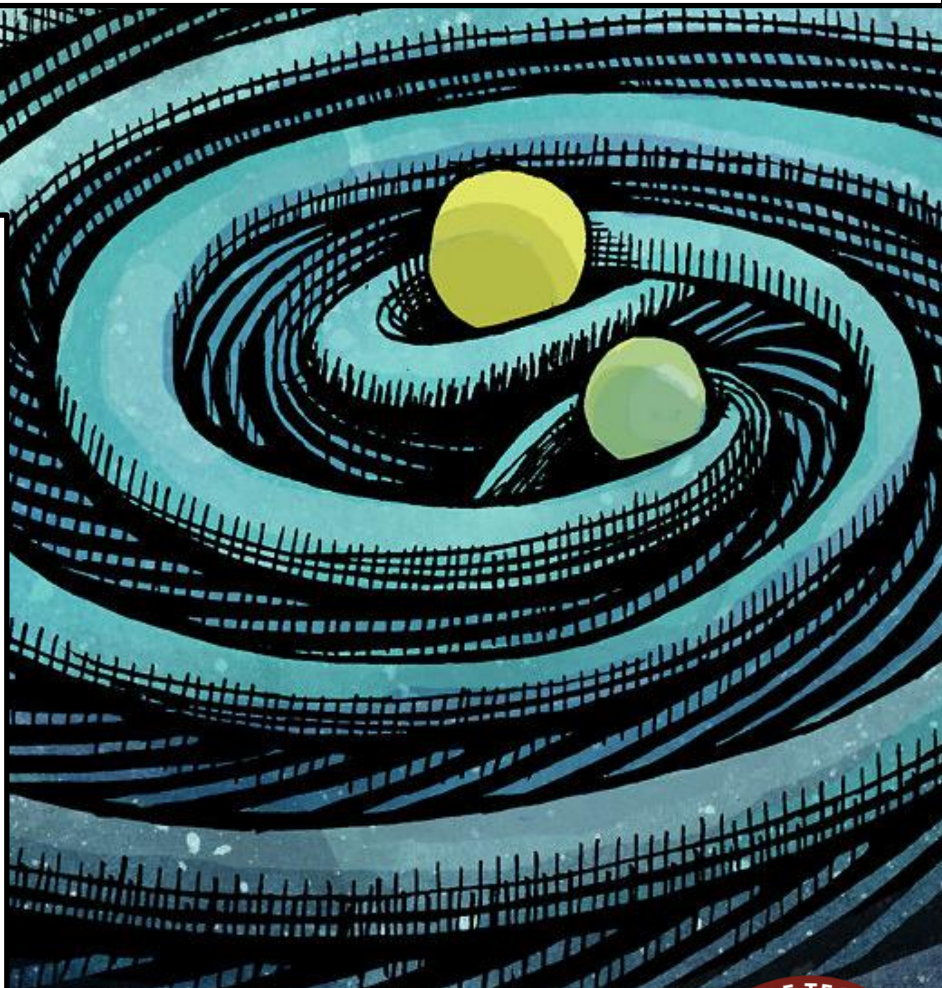

Abstract: LIGO's first-ever detection of gravitational waves was consistent with black holes moving in circular orbits as predicted by General Relativity. The future generation of gravitational waves detectors like LISA will be able to detect signals from binaries that have very small orbital eccentricities when they enter the low-frequency band of such detectors. The gravitational waveform from such systems, modelled using Post-Newtonian methods are used as template banks to match the signals from the detector. The work follows analytical modelling of gravitational waveforms of eccentric binary black holes in the frequency domain. Post-Newtonian waveform analytic models in the frequency domain admit simple structure, allowing computationally efficient data analysis. We use previously computed time-domain waveforms for compact binaries in eccentric orbits to calculate the frequency domain waveform amplitude under small-eccentricity approximation. The gravitational waveform of a merging stellar-mass binary is described at leading order by a quadrupolar mode. However, the complete waveform includes higher-order modes. The following work consists of these higher harmonics in the frequency domain waveform.

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Motivation and Modelling Compact Binaries

Eccentric compact binaries

Gravitational waves carry energy from in spiralling binaries and circularise the orbits. Hence usually the inspiralling compact binaries are modelled for quasi-circular orbits. There is an increased interest in inspiralling binaries with very small orbital eccentricity when they enter the frequency band of Gravitational wave detectors (Fig[2]). If the eccentricity is not accounted for, it can cause a significant systematic error in mass parameters of an inspiralling binary. The orbital parameters are described in Fig[1]. The orbital variables (r, ϕ) and their derivatives are expressed as function of the mean anomaly u and evolved through time.

$$r_N = a_r(1 - e\cos u) \qquad \phi_N = \lambda_N + v - u + e\sin u$$

$$\dot{r}_N = \frac{\xi^{1/3} e \sin u}{(1 - e\cos u)} \qquad \lambda_N = n(t - t_0) + C_\lambda$$

$$l_N = n(t - t_0) + C_l = u - e\sin u$$

[Moore et. al.(2016)]

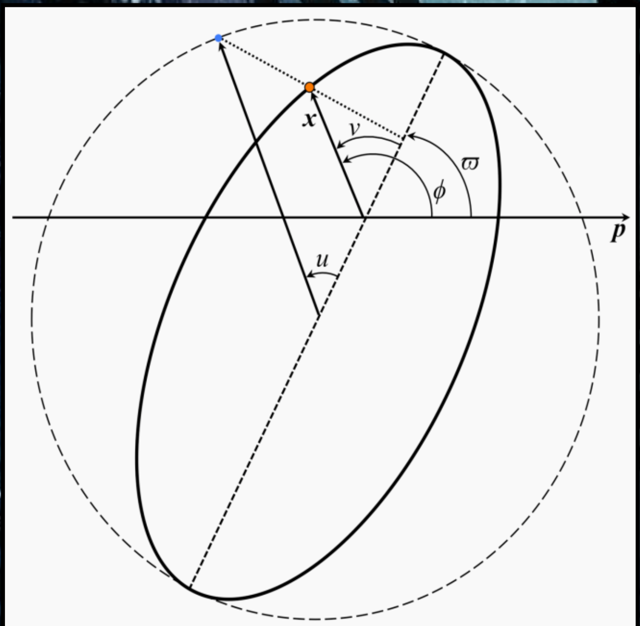

Figure [1]: The various orbital elements *[Moore et. al.(2016)]*

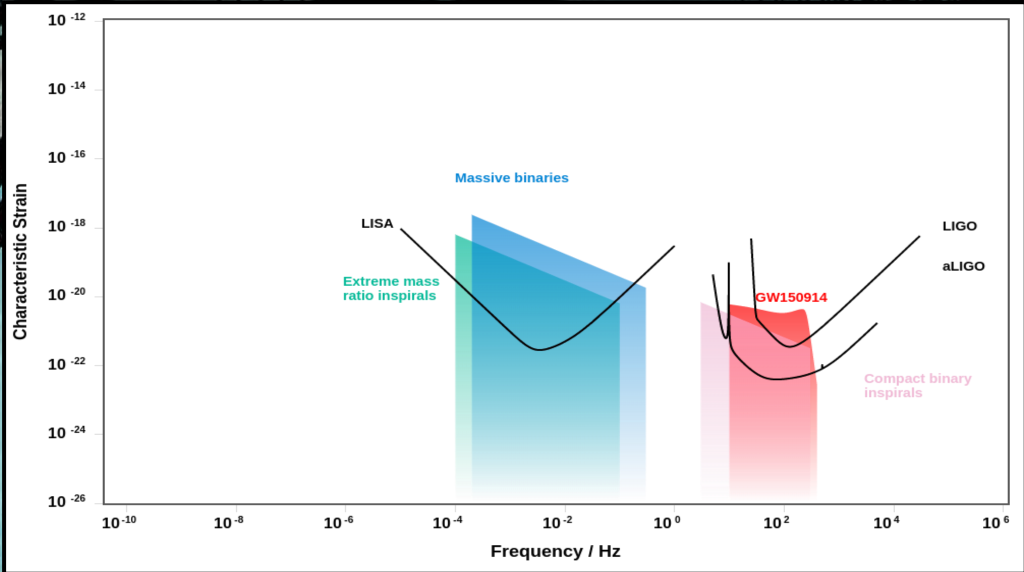

Figure 2: The sensitivity curves for LIGO and LISA detectors. [http://gwplotter.com]

Post-Newtonian Approximation

In Post-Newtonian approximation, the equations of general relativity take the form of the familiar Newtonian two-body equations of motion, in the limit $\frac{v}{c} \to 0$, known as *the weak field limit*. A correction of $(v/c)^n$ to the Newtonian equations counts as an $\frac{n}{2}$ order in the PN expansion. For example, the two-body equation of motion becomes:

$$\frac{dv}{dt} = -\frac{GM}{r^2}\left(1 + \frac{\alpha_{1PN}}{c^2} + \frac{\alpha_{1.5PN}}{c^3} + \frac{\alpha_{2PN}}{c^4} + \cdots\right)$$

At each PN order we unravel new physics beyond the Newtonian regime. The PN parameter x is defined as $x = v^2/c^2$. The evolution of the orbital phase and the separation are determined by solving the Energy-Balance equation:

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$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{x^{3/2}}{M}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{F(x)}{dE/dx}$$

The solution provides us the amplitude and the phase required for construction the two polarization states $(h_+, h_\times)$ of gravitational wave. These are decomposed into a multipole expansion where h^{lm} denotes the various “modes” of the wave.

$$h_+ - ih_\times = \sum_{l=2}^{\infty}\sum_{m=-l}^{l} h^{lm} Y^{lm}_{-2}(\Theta, \Phi)$$

[Boetzel et. al.(2016)]

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For black hole binaries characterized by a low mass ratio ($q \le 4$) and a total mass less than 100 solar mass, the $(l = 2, m = \pm 2)$ modes dominate.

Stationary Phase Approximation (SPA) calculations

Post Newtonian time domain expressions

The time-domain inputs [1] are expressed in terms of Post-Newtonian order with the leading order being the Newtonian term or the 0 PN order. We convert the Fourier transform of these-time domain expressions which admits a simple analytic structure.

$$h^{lm} = \frac{8\,G\,m\nu}{c^2 R}\,x\sqrt{\frac{\pi}{5}}\,e^{-i\,m\psi}\,H^{lm}$$

$$H^{22}_{Newt} = \bar{e}\left\{\frac{1}{4}e^{-i\xi} + \frac{5}{4}e^{i\xi}\right\},$$

[Boetzel et. al.(2016)]

$$H^{22}_{1PN} = x\left(-\frac{107}{42} + \frac{55\nu}{42} + \bar{e}\left\{e^{-i\xi}\left[-\frac{257}{168} + \frac{169\nu}{168}\right] + e^{i\xi}\left[-\frac{31}{24} + \frac{35\nu}{24}\right]\right\}\right),$$

$$H^{22}_{1.5PN} = x\left(2\pi + \bar{e}\left\{e^{-i\xi}\left[\frac{11\pi}{4} + \frac{27i}{2}ln(3/2)\right] + e^{i\xi}\left[\frac{13\pi}{4} + \frac{3i}{2}ln(2)\right]\right\}\right),$$

$$H^{22}_{2PN} = x^2\left(-\frac{2173}{1512} + \cdots + \bar{e}\left\{e^{i\xi}\left[-\frac{2155}{252} + \ldots\right] + e^{-i\xi}\left[-\frac{4271}{756} + \ldots\right]\right\}\right),$$

$$H^{22}_{2.5PN} = x^{5/2}\left(-\frac{107\pi}{21} + \cdots + \bar{e}\left\{e^{i\xi}\left[-\frac{-9i}{2} + \ldots\right] + e^{-i\xi}\left[-\frac{-27i}{2} + \ldots\right]\right\}\right),$$

$$H^{22}_{3PN} = x^3\left(\frac{27027409}{646800} + \cdots + \bar{e}\left\{e^{i\xi}\left[-\frac{55608313}{1058400} + \ldots\right] + e^{-i\xi}\left[-\frac{219775769}{1663200} + \ldots\right]\right\}\right),$$

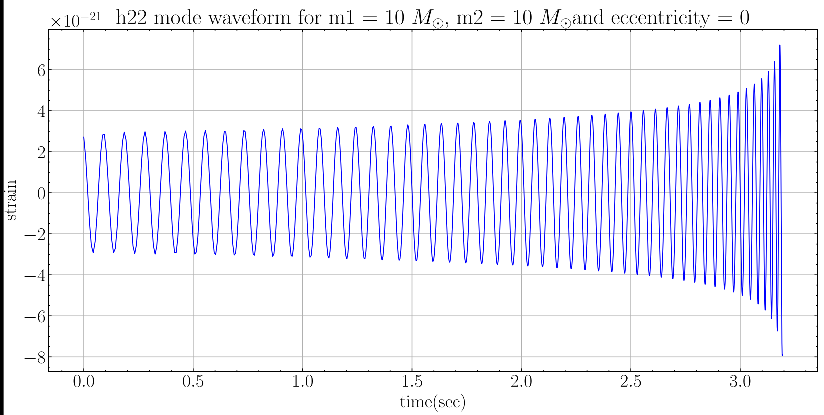


Figure 3: The 22 mode 3PN order time domain expression plot.

Conversion to frequency domain

The time domain expressions are now converted to frequency domain using a Fourier transform,

$$\tilde{h}(f) = \int_{-\infty}^{\infty} h^{lm}(t)e^{2\pi i f t}\,dt$$

Such an integral can be computed using the stationary phase approximation(SPA) [2]. For large values of f, the integrand oscillates rapidly and causes large cancellations when integrated.

$$I(f) = \int_a^b \Lambda(t)\mathrm{e}^{\mathrm{i}f\varphi(t)}\,\mathrm{d}t$$

This integral is dominated by times when the phase is stationary($\dot{\varphi} = 0$). Hence, near the stationary point I can Taylor expand φ as

$$\varphi(t) \approx \varphi(t_0) + \frac{1}{2}\ddot{\varphi}(t_0)(t-t_0)^2 + \cdots$$

Substituting this into the integral results,

$$\tilde{h}(f) \approx \Lambda(t_0)e^{if\varphi(t_0)}\int_{-\infty}^{\infty} e^{\frac{if}{2}\ddot{\varphi}(t_0)(t-t_0)^2}\,dt$$

$$\tilde{h}(f) \approx \Lambda(t_0)\sqrt{\frac{2\pi}{\ddot{\varphi}(t_0(f))}}\,\mathrm{e}^{\mathrm{i}(f\varphi(t_0(f))-\pi/4)}$$

Where $\dot{\varphi} = 0$ (stationary point).

For the quasicircular case, the stationary condition gives us,

$$2\pi f = m\dot{\phi}(t_0) = 2\pi\, mF(t_0)$$

Where ϕ is related to the instantaneous orbital frequency $F(t)$. Thus we get mth harmonic frequency as,

$$f = m\,F(t_0)$$

For eccentric time domain expression we have additional nonlinear terms that enter the Fourier transform integral. To treat these terms, we do an approximation of the phase as $\phi = l + dl$.

Where l is the mean anomaly. At the Newtonian order ϕ and l are equivalent. $\mathrm{d}l$ is the correction term. Using $\phi = l + dl$ we rewrite the integrals as,

$$e^{\mp k dl}\int_{-\infty}^{\infty} e^{-im\phi}e^{\pm k\phi}e^{2\pi i f t}\,dt$$

The stationary condition changes as,

$$f = (m \mp k)\,F$$

In the eccentricity $e \to 0$ limit, you get the circular limit. Here k depends on the order of eccentricity of the time-domain expression.

Results and Future Work

Frequency domain expression for higher modes

Using the SPA method, we find a general form for the expression in frequency domain. The frequency domain is expanded in powers if *v* to 3PN order. The coefficients involved in the expressions are worked out for all modes from l=2 to l=5.

$$h_{\ell m} = \frac{G^2 M^2 \pi}{c^5 R}\sqrt{\frac{2\nu}{3}}\ \sum_{k=-1}^{k=1} \sum_{n=0}^{n=6} C^{\ell m}_{kn}\ v_k^{n-7/2}\ e^{i(m-k)\Psi_{SPA}}\ e^{-ikdl_k}\ e^{-imW_k}$$

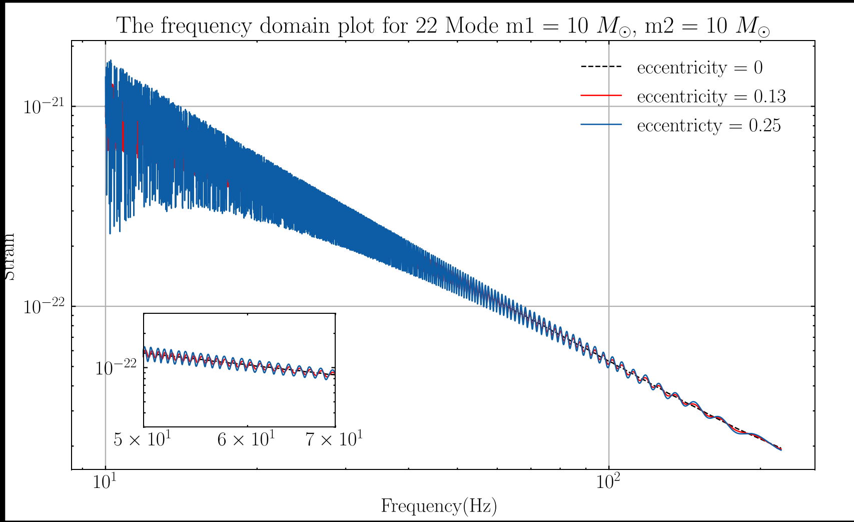
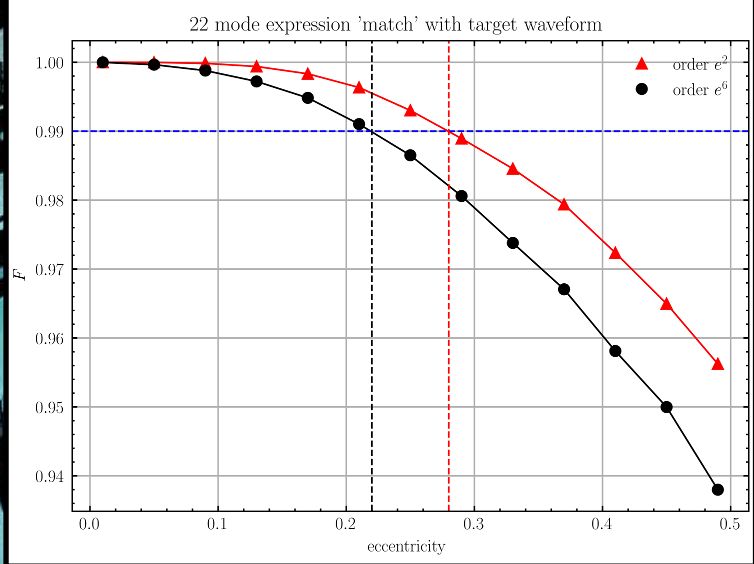


Figure 4: A plot of the 22 mode in frequency mode for various initial eccentricity. We see oscillations in the plot that amplifies when eccentricity increases

Figure 5: The “match” plots..



Comparison with the most accurate waveform

The most accurate 3PN expression is when we include the full expression for the eccentricity-induced oscillatory phase term *W* with eccentricity expanded to $O(e^6)$. Using this as a target waveform, we calculate the ”match” for waveforms with just a leading order *W* phase term. The “match” is calculated as

$$F(h_1, h_2) = \frac{(h_L|h_2)}{\sqrt{(n_1|h_1)(h_2|h_2)}}, (h_1|h_2) = \int_{f_0}^{f_{iso}} \frac{h_1^*(f) h_2(f)}{s_n(f)} df$$

Figure 5 shows the “match” plots using the full eccentricity expanded to $O(e^6)$ as the target waveform. The red plot is a match for eccentricity expanded to second order with the target with a leading-order phase term *W*. The black plot is for the same phase but for eccentricity expanded to order 6. We see a match of 99 % for eccentricity around 0.22 and 0.28.

Acknowledgment and Future work

My work was done under the guidance of Chandra Kant Mishra, IIT Madras.

We aim to compare our analytic waveforms with numerical FFTs for accuracy. We further aim to incorporate spin effects in amplitude and phase for frequency-domain waveforms.

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